A Structural Model of Credit Risk for Illiquid Debt

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Abstract

In this paper, we develop a structural credit risk model that relies on cash flow data to derive credit risk metrics. The model is useful for illiquid assets for which a time series of prices is not observable. Our methodology is designed to require a parsimonious dataset of observable inputs, and provides a clear link between an asset's fundamental characteristics and its risk profile. The model is flexible enough to value debt instruments with path-dependant cash flows, such as mortgages and floating rate loans, and can incorporate various debt covenants, such as debt refinancing, and restructuring options, as well as cash sweeps, dividend lockups, and reserve accounts. The implementation of the model is illustrated with project finance debt, which is highly illiquid, and suffers from a serious lack of price data. We show that the dynamics of the debt service cover ratio (DSCR) along with the debt repayment profile and the debt covenants is sufficient to implement our credit risk model. For reasonable parameter values of the DSCR dynamics, the model reproduces stylised empirical regularities regarding the probabilities of default for two generic types of infrastructure projects.

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Traditional structural models of credit risk assume that firm value is observable, and that default occurs when firm value falls below the value of outstanding debt. This remains valid if firms can raise new cash to make their debt payments, as long as the present value of their future cash flows exceeds the value of their debt. In turn, this implies that the value of firm is public information, and that the firm has unrestricted access to capital markets. For publicly traded firms, these assumptions can seem reasonable. The market value of such firms can be obtained from the market value of their publicly traded equity claims, which is public information, and their access to capital markets allows them to raise funds to repay any outstanding debt, as long as the firm value exceeds the total value of debt.

However, this is not the case for private firms and individuals. While they do have income that lies in the future, the present value of these future cash flows is not public information as the

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claims against them are not publicly traded, and private firms and individuals cannot always borrow against their future income, as their access to public capital is limited. Private firms do not have access to public capital markets, and may fail to borrow against their future cash flows due to asymmetric information (Sharpe (1990); Whited (1992)). Similarly, mortgage holders, and individuals in general, cannot always borrow against their future income to make their mortgage payments, due to moral hazard (Hubbard and Judd (1987); Hansen and Imrohoroğlu (1992)). The most topical example is infrastructure project finance, in which the firm is structured as stand-alone special purpose entity financed wth private debt and contractually barred from raising additional borrowing.

In such cases, traditional credit risk models cannot be directly applied to value such private illiquid debt.

In this paper, we develop a structural credit risk model that does not assume a frictionless access to capital. Our model derives credit risk metrics using the free cashflow available for debt service (CFADS), which is more easily observable for private illiquid instruments, and may differ across firms with identical firm values but different access to capital. For a given firm value, the firms with better access to capital markets would be able to raise more cash and have higher CFADS. Default is triggered when the CFADS in a given period falls below the scheduled debt service in that period.

Thus relating default and credit risk metrics to the CFADS gives a higher degree of generality to the model, which can incorporate any restrictions on the access to capital markets, and allows our model to be applicable to private firms and individual borrowers. Our model is flexible enough to take into account path dependent cash flows, various debt covenants, such as dividend lockup, technical default, reserve accounts, as well as debt refinancing and restructuring options. Thus our model provides a clear link between underlying cash flow risk (revenue risk) profile, debt covenants, and the credit risk profile.

Starting from the first structural model developed in Merton (1974), which only allows for debt with a single repayment at the maturity, structural models have evolved to incorporate complex capital structures (Jones, Mason, and Rosenfeld (1985), Jones, Mason, and Rosenfeld (1984), Black and Cox (1976)), stochastic interest rates (Longstaff and Schwartz (1995), Heston (1993), David C. Shimko, Naohiko Tejima and Deventer (1993)), stochastic volatility (Heston (1993), Guo, Jarrow, and Zeng (2009)), jump diffusion process (Delianedis and Geske (2001), Zhou (1997)), incomplete information (Bellalah (2001), Duffie and Lando (2001), Guo, Jarrow, and Zeng (2009), Giesecke (2004), Giesecke (2006)), exogenous (Black and Cox (1976), Longstaff and Schwartz (1995)) and endogenous (Leland (1994), Anderson and Sundaresan (1996)) default threshold, and strategic debt service (Anderson and Sundaresan (1996), Mella-Barral and Perraudin (1997)).

Our model combines several elements of these existing structural models, such as general debt repayment profiles, stochastic interest rates, and exogenous and endogenous default thresholds. Its main contribution to the existing literature is to specify a structural model directly in terms of the firms' cash flows, instead of asset values, allowing for the integration of various debt covenants, and incorporating approximate arbitrage bounds.

Specifying the model in terms of cash flows allows us to use observable quantities to calibrate a model of illiquid debt credit risk, typically issued by private firms or individuals, the market value of which is not available.

We show that understanding the dynamics of the firm's debt service cover ratio (DSCR), along with the debt repayment profile and the debt covenants, which are observable, is sufficient to implement a fully-fledged structural model.

Incorporating debt cash-flow related covenants allows taking into account the effects of 'credit events' or covenant breaches which, while they do not necessarily correspond to a "hard" default (of payment), can nevertheless affect the risk profile of debt.

Incorporating approximate arbitrage bounds allows determining reasonable valuation bounds for illiquid debt instruments, which may be valued differently by different investors with heterogeneous preferences.

We illustrate the implementation of this approach in the case project finance (PF) debt, which is highly illiquid, and includes several debt covenants that are not taken into account in traditional credit risk models. The DSCR is routinely monitored in such transactions.

For reasonable parameter values of the DSCR dynamics, the model reproduces stylised empirical regularities regarding the probabilities of default for two generic types of infrastructure projects.

1. Structural Credit Risk Model

In this section, we formalise our model for a generic debt contract, which may include several debt covenants, using a stochastic model of the borrower's free cash flow available for debt service, which could correspond to the CFADS of a firm or an individual project, or to the income of an individual, all of which are observable. Thus, the model can directly relate credit risk metrics to the observable cash flow process. The model mainly consists of the following components:

1. A cash flow model that projects cash flows to different stakeholders given project characteristics and debt covenants;
2. A model to risk-neutralise the cash flow distribution to incorporate investors' risk preferences;
3. A Black Cox decomposition to determine present value of the debt.

Next, we outline each step of the valuation framework.

1.1. Cash Flow Dynamics

The first step in our valuation framework is to project the Cash Flow Available for Debt Service (CFADS) in every state of the world. To avoid any issues due to scale dependence of the CFADS, we write our cashflow model in terms of the Debt Service Cover Ratio (DSCR), which is related to the CFADS in a deterministic manner according to

$$CFADS_t = DSCR_t \times \mathrm{DS}_t^{\mathrm{BC}} \tag{1}$$

with $\mathrm{DS}_t^{\mathrm{BC}}$, the base case debt service defined at financial close. The same relationship holds in expectation.

In other words, as long as the base case debt service is known, we can reduce the question of modelling the free cash flow to that of the dynamics of $DSCR_t$, which, in its general form, can be written as

$$\frac{d(DSCR_t)}{DSCR_t} = \mu_t + \sigma_t dW_t, \tag{2}$$

where μ_t and σ_t may be stochastic. Thus the DSCR provides us a scale independent quantity that is related to the CFADS in a deterministic manner, and is directly comparable across debt investments of various sizes.

1.1.1. Default Point

In structural models of standard corporate debt, default is generally modelled as crossing a threshold point below which the total value of the firm's assets is less than its short and medium term liabilities. This is because as long as the total value of the firm is higher than its near term liabilities, equity holders can raise more cash by issuing new equity or debt, and satisfy their current debt obligations. However, as discussed before, this is not always the case due to borrowing restrictions.

Therefore, we define default directly as the inability to service debt, or to respect the debt contract more generally, because debt contracts often impose other obligations on the borrower in addition to debt repayment and create the possibility of *technical* defaults. Examples of such technical defaults may include inability to maintain sufficient collateral, DSCR, or liquidity ratios (Beneish and Press (1993); Chava and Roberts (2008)). Here, we specify our default point in terms of the DSCR as

$$\text{Default}_t \iff DSCR_t \equiv \frac{CFADS_t}{\text{DS}_t^{\text{BC}}} < 1.x. \tag{3}$$

A a "hard" default, i.e. an actual default of payment, occurs when the DSCR falls below 1.0, and a technical default may be triggered when the DSCR falls below some level, say 1.05, specified in the debt contract.

1.1.2. Distance to Default

Knowledge of the dynamics of $DSCR_t$ is sufficient to derive the firm's "distance to default", which is directly related to the probability of default. For example, in the Merton model (Merton (1974)), the firm's assets are assumed to follow a log-normal process with a constant mean and volatility, and the physical probability of default is given by

$$p(t,T) = N\left(\frac{\ln(\frac{A_t}{D}) + (\mu - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right), \tag{4}$$

where $p(t,T)$ is the cumulative probability of default between time t to T, A_t is the value of the firm's assets at time t, D is the default threshold, and μ and σ are the mean return and volatility of firm's assets.

Drawing from the Merton model, the KMV model (Crosbie and Bohn (2003)) defines the negative of the quantity inside the brackets as the Distance to Default (DD)

$$DD_T = -\frac{\ln(\frac{A_t}{D}) + (\mu - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \tag{5}$$

The default probability is the area under the distribution above the DD point

$$p(t,T) = N(-DD_T). \tag{6}$$

The distance to default can be approximated as (McNeil, Frey, and Embrechts (2005); Crosbie and Bohn (2003))

$$\text{Distance to Default} = \frac{[\text{Market value of assets}] - [\text{Default point}]}{[\text{Market value of assets}].[\text{Asset volatility}]}, \tag{7}$$

where the asset volatility is the standard deviation of the annual percentage change in the asset value.

The KMV model premises that the DD is a sufficient statistic to arrive at a rank ordering of default risk, where the numerator in (5) expresses the firm's financial leverage or *financial risk*, while the denominator reflects its *business risk*.

Expressing default in terms of the free cash available for debt service, Distance to Default at time t can be defined as

$$DD_t = \frac{CFADS_t - \mathrm{DS}_t^{\mathrm{BC}}}{\sigma_{CFADS_t} CFADS_t} \tag{8}$$

Using the definition of $DSCR_t$ in (1), the above expression can be written as:

$$DD_t = \frac{1}{\sigma_{CFADS_t}}(1 - \frac{1}{DSCR_t}) \tag{9}$$

The above can be re-written as a sole function of $DSCR_t$ by expressing the volatility of $CFADS_t$ as a function of that of $DSCR_t$ (as shown in 5.1)

$$DD_t = \frac{1}{\sigma_{DSCR_t}} \frac{\mathrm{DS}_{t-1}^{\mathrm{BC}}}{\mathrm{DS}_t^{\mathrm{BC}}}(1 - \frac{1}{DSCR_t}) \tag{10}$$

where σ_{DSCR_t} is the standard deviation of the annual percentage change in the $DSCR$ value.

1.1.3. Cashflow Waterfall

Once, we have a model for the DSCR, we simulate $DSCR_t$ for every period over the relevant horizon, and compute the corresponding $CFADS_t$. The cashflows to different stakeholders can then be projected by constructing a cashflow waterfall using debt covenants in the debt contract. The cashflow waterfall distributes the CFADS across different stakeholders by following the seniority of different claims specified in the debt contract. For instance, we start with the senior debt payment, then move on to any reserve account requirements for senior debt, then to mezzanine debt payment, then to any cash sweep requirements, and finally pay the remaining cash to equity holders. We describe how to construct the cashflow waterfall in more detail in Section 2.

1.2. Risk neutralisation

The aim of the risk-neutralisation is to incorporate investors' risk preferences in the valuation model. In structural credit risk models, this is done by risk-neutralising the distribution of the underlying stochastic variable (firm value). The risk-neutralisation incorporates investors' risk aversion by discounting risky cash flows, and once the risk-neutral distribution of firm value has been obtained, the securities issued by the firm can be valued by discounting their cash flows at the riskfree rate.

In the Merton model, the mapping between risk-neutral and physical probabilities of default is given by (Kealhofer, 2003)

$$q(t,T) = N\left(N^{-1}[p(t,T)] + \lambda_T\right), \tag{11}$$

where $q(t,T)$ is the risk neutral cumulative probability of default between time t and T, and $\lambda_T = \frac{1}{\sqrt{T}} \int_t^T \frac{\mu(t') - r(t')}{\sigma(t')} dt'$ is the required Sharpe ratio over this horizon. When μ, r, and σ are constant, the price of risk over horizon T becomes $\lambda_T = \frac{\mu - r}{\sigma}\sqrt{T}$.

The corresponding risk neutral distribution for DSCR can be written as (Wang (2002))

$$F^*(\mathrm{DSCR}_T) = N\left(N^{-1}[F(\mathrm{DSCR}_T)] + \lambda_T\right), \tag{12}$$

where $F(\mathrm{DSCR}_T)$ and $F^*(\mathrm{DSCR}_T)$ are the physical and risk-neutral distributions of DSCR_T.

If the physical distribution $(F(x))$ is normal $(X \sim N(\mu,\sigma))$, or lognormal $(ln(X) \sim N(\mu,\sigma))$, then the risk neutral distribution $(F^*(x))$ follows the same distribution (normal or lognormal) with a shifted mean $\mu - \lambda\sigma$. Hence, the risk neutral distribution of the DSCR would the same as the physical distribution of the DSCR but with a shifted mean.

1.2.1. Decomposition of Risk Into Traded and Non-Traded Components

The price of risk for an asset whose cash flows can be replicated using traded securities, is determined by the price of risk of the replicating portfolio, according to no-arbitrage principle. However, for assets whose cash flows are not spanned, i.e. cannot be perfectly replicated, by traded securities, the price of risk cannot be uniquely determined using the prices of traded securities. This is often the case with private illiquid instruments, which are often weakly correlated with publicly traded securities. In order to determine the required Sharpe ratio, λ, for such assets, we can decompose the underlying CFADS process into a component that is spanned by traded securities, and a component that is not (Froot and Stein (1998)). That is, we write the current period's CFADS as

$$CFADS_{t-1} = CFADS^T_{t-1} + CFADS^N_{t-1},$$

where $CFADS^T_{t-1}$ represents the component of CFADS generated by the replicating portfolio of traded securities, and $CFADS^N_{t-1}$ represents the components of the CFADS not generated by the replicating portfolio. Then, we can write the mean return on CFADS as (see 5.2)

$$\mu = w^T_{t-1}\frac{\sigma^T}{\sigma}\lambda^T + w^N_{t-1}\frac{\sigma^N}{\sigma}\lambda^N,$$

where we have defined $w^{T(N)} = \frac{CFADS^{T(N)}_{t-1}}{CFADS_{t-1}}$, and $\lambda^{T(N)} = \frac{\mu^{T(N)} - r}{\sigma^{T(N)}}$.

This separation of risks serves two main purposes:

1. The required prices for hedgable risks can be set equal to the premium earned by the traded portfolio, to prevent arbitrage.
2. The required prices for unhedgable risks would lie in an approximate arbitrage band, as investors with heterogeneous preferences may be willing to pay different prices for similar assets.

The decomposition of CFADS into traded and untraded components is an empirical task, and can only be done once sufficient data is available to estimate the correlations between the CFADS and cash flows on traded securities. Thus, in the model implementation detailed in section 3, we assume that the CFADS process is completely uncorrelated with traded securities, and hence the w^T is zero.

1.2.2. Choice of Bounds on Required Risk Premium

Before proceeding with the valuation model, we discuss our choice of bounds of the required Sharpe ratio λ. We argue that the investors' Sharpe ratios would lie in a band between 0 and 2.

Indeed, annualised Sharpe ratios for market indices typically fall below 1.0, and the largest Sharpe ratios are often exhibited by hedge funds. Even for high performing hedge funds, the only instances where the Sharpe ratio may exceed 2.0 are when the returns are not normally distributed (Kat and Brooks (2001)). Non-normal distributions exhibit higher moment risks, such as negative skewness, high kurtosis, and the Sharpe ratio (which only takes into account the first two moments) can underestimate the riskiness of such investments.[3]

Since we assume normal distribution for the underlying risk in our examples,[4] we argue that if an asset offered Sharpe ratios above this upper limit of 2, they would become too attractive, and such loans would soon disappear from the market. Therefore, in equilibrium, the Sharpe ratios for assets would lie between 0 and 2.

Theoretical justification for bounds on risk/reward ratios is discussed in Cochrane and Saa-Requejo (2000); Bernardo and Ledoit (2000). Cochrane and Saa-Requejo (2000) show that even with high levels of risk aversion and volatility in future levels of consumption, Sharpe ratios do not exceed 1.72. Hence, our choice of an upper limit of 2.0 seems justified from both an applied and a theoretical perspective.

1.3. Black Cox Decomposition

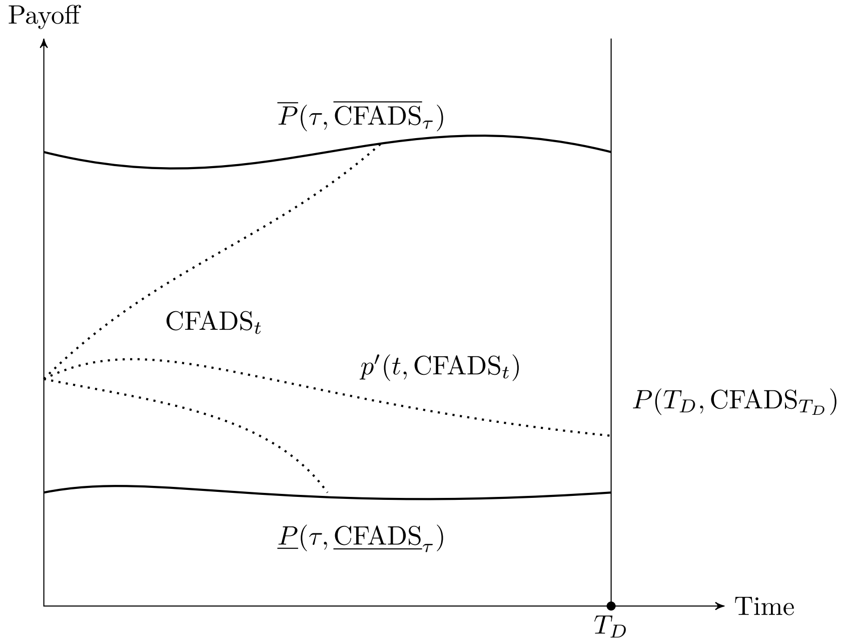
The Black-Cox decomposition (Black and Cox (1976)) was devised to value corporate securities when firms can reorganise. However, the original model assumes that the reorganisations happen when the total value of the firm reaches a lower or an upper boundary, whereas our model is driven by cashflow dynamics. Therefore, we modify the Black-Cox decomposition to take into account this difference. We define 4 payout functions, as illustrated in Exhibit 1:

1. $P(T_D, \text{CFADS}_{T_D})$: final payment at the maturity of the contract. (We use T_D to refer to the maturity of the debt contract, which may be different from the maturity of the project denoted earlier by T.)
2. $\underline{P}(\tau, \text{CFADS}_\tau)$: the value of the security if the CFADS reaches the lower boundary at time τ.
3. $\overline{P}(\tau, \text{CFADS}_\tau)$: the value of the security if the CFADS reaches the upper boundary at time τ.

[3]For example, the Long-Term Capital Management (LTCM) exhibited a Sharpe ratio of 4.35 before its demise in 1998 (Lux (2002)). However, as is now well known, the hedge fund was exposed to some extreme risks, and the return distribution was highly non normal.

[4]For non-normal distributions, the bounds can be specified using other risk reward ratios, such as the gain-loss ratio introduced by Bernardo and Ledoit (2000).

Exhibit 1: Black-Cox decomposition at one point in time. $\overline{P}(\tau, \overline{\text{CFADS}}_\tau)$ is the payout function if CFADS hits the upper boundary, $\underline{P}(\tau, \underline{\text{CFADS}}_\tau)$ is the payout function if CFADS hits the lower boundary, $P(T_D, \text{CFADS}_{T_D})$ is the payout function at the maturity of the debt, and $p'(t, \text{CFADS}_t)$ is the payout function before CFADS hits any of the boundaries or reaches maturity.



4. $p'(t, \mathrm{CFADS}_t)$: the payments made by the debt security until the maturity or reorganisation.

The total value of a security is the expected present value of the sum of these 4 payout functions under the risk-neutral measure, discounted at the risk-free rate.

1.3.1. Restructuring

We consider the lower reorganisation boundary as the default boundary, where creditors have the right to restructure their debt. The value of debt upon restructuring is determined by how the debt is restructured, which in itself may be modelled as an outcome of strategic bargaining between debt and equity holders. However, our aim in this paper is not to provide a model for optimal restructuring, but only a credit risk model that can take into account such restructuring.[5]

Hence, in this setting, we assume that the value of debt upon restructuring is exogenously given, and leave the task of modelling the outcome of restructuring. We denote the debt value upon restructuring, which provides the debt value at the lower boundary, by $\underline{P}(\tau, \mathrm{CFADS}_\tau)$.

1.3.2. Refinancing

To model the outcome of reorganisations at the upper boundary (refinancing), we make a few simplifying assumptions. Firstly, we ignore the effects of market conditions such as the level of interest rates, demand for illiquid debt etc., and assume that the refinancing does happen as soon as the CFADS hits a predetermined boundary. In other words, we assume that as soon as the $CFADS_t$ crosses a certain threshold, the debt's level of riskiness decreases sufficiently to justify a reduction in the cost of debt, irrespective of market conditions. Secondly, we assume that upon refinancing, the amount of debt outstanding is paid in full along with any costs or penalties imposed by debt covenants.

In the Black-Cox decomposition discussed above, the value of debt at the upper reorganisation boundary is given by

$$\overline{P}(\tau) = (1 + c) \left[\sum_{i=\tau}^{T_D} e^{-rate(i-\tau)} \mathrm{DS}_i \right], \tag{13}$$

where c is the refinancing costs, $rate$ is the original IRR of the loan, and DS_i^{BC} is the scheduled debt payment at time i.

1.3.3. Putting It All Together: Total Value of Debt

In the Black-Cox decomposition, the task of valuing a security largely reduces to identifying the four payout functions of the security, and then determining the present value of those payouts, which is described below.

[5]We propose such a model of private debt restructring in a forthcoming paper.

Using $\kappa(.)$ to denote the interval $\left(\underline{\text{CFADS}}(.), \overline{\text{CFADS}}(.)\right)$, we can write the value, $h_1(V_t, t)$, of the first payout function as

$$
\begin{aligned}
h_1(V_t, t) &= E\left[e^{-r_{T_D,t}(T_D-t)} P(T_D, \text{CFADS}_{T_D})\right] \\
&= e^{-r_{T_D,t}(T_D-t)} \int_{\kappa(T)} P(T_D, \text{CFADS}_{T_D}) dF^*,
\end{aligned} \tag{14}
$$

where dF^* is the probability of CFADS_T falling between the two boundaries at time T_D.

The value of the fourth component is obtained by summing over all the payouts from time t to T_D

$$
h_4(V_t, t) = \int_t^{T_D} e^{-r_{s,t}(s-t)} \times \left[\int_{\kappa(s)} p'(\text{CFADS}_s, s) dF^*(\text{CFADS}_s, s)\right] ds. \tag{15}
$$

In order to determine the contribution of the second and the third components, one needs to determine the hitting times (times at which the CFADS hits a boundary), and the value of the debt security at the corresponding boundary. We denote the first time CFADS hits the lower boundary by $T_{\underline{\text{CFADS}}}$, and the first time CFADS hits the upper boundary by $T_{\overline{\text{CFADS}}}$. Further, let $F^*_{T_{\underline{\text{CFADS}}}}$ denote the risk neutral probability density function of the first passage time $T_{\underline{\text{CFADS}}}$, and $F^*_{T_{\overline{\text{CFADS}}}}$ denote the risk neutral probability density function of the first passage time $T_{\overline{\text{CFADS}}}$. We can then write

$$
h_2(V_t, t) = \int_t^T e^{-r_{T_{\underline{\text{CFADS}}},t}(T_{\underline{\text{CFADS}}}-t)} \times \underline{P}(\text{CFADS}_{T_{\underline{\text{CFADS}}}}, T_{\underline{\text{CFADS}}}) dF^*_{T_{\underline{\text{CFADS}}}} \tag{16}
$$

$$
h_3(V_t, t) = \int_t^T e^{-r_{T_{\overline{\text{CFADS}}},t}(T_{\overline{\text{CFADS}}}-t)} \overline{P}(\text{CFADS}_{T_{\overline{\text{CFADS}}}}, T_{\overline{\text{CFADS}}}) dF^*_{T_{\overline{\text{CFADS}}}}. \tag{17}
$$

$F^*_{T_{\underline{\text{CFADS}}}}(T_{\underline{\text{CFADS}}})$ and $F^*_{T_{\overline{\text{CFADS}}}}(T_{\overline{\text{CFADS}}})$ can be evaluated either analytically or numerically depending on the stochastic process followed by the $CFADS_t$.

The total value of the security is then

$$
V^S(V_t, t) = \sum_{i=1}^{i=4} h_i(V_t, t), \tag{18}
$$

where $h_i(V_t, t)$ is the value of the security at time t from the i^{th} payout function, and $V^S(V_t, t)$ is the total value of the security at time t, and V_t is the value of all future cash flows at time t.

It should be stressed that in the case of a restructuring option, all payout functions are not determined by the original debt contract. In particular, the payout at the lower boundary (default threshold) is not specified in the original contract, but is determined by the exercise of the debt restructuring option.

Nevertheless, once the payout functions have been determined, we can discount the security's payouts to determine its present value. The appropriate discount rate in the case of risk neutral valuation is the risk free rate, as the effects of risk preferences have already been incorporated in the risk neutral probability measure.

That is, one can simply compute the expected payouts of security at every point in its life under the risk neutral measure, and then discount them at the risk free rate to determine its fair value.

2. Algorithm

In this section, we provide an algorithm for the numerical implementation of the theoretical model, as illustrated by Exhibit 2, assuming some common debt covenants such as reserve accounts, cash sweeps, refinancing, and debt restructuring. Other debt covenants such clawback provisions can also be implemented using within this setup. The main steps in implementing the framework are

1. Obtain the base case debt schedule;
2. Select a model for DSCR distribution;
3. Determine the CFADS distribution using the DSCR distribution and the base case debt schedule;
4. Risk-neutralise the CFADS distribution: Select a required Sharpe ratio, and shift the original DSCR (or CFADS) distribution accordingly;
5. Obtain debt covenants: Debt covenants may contain reserve accounts, cash sweeps and clawback provisions etc. and include the technical default threshold: the threshold below which lenders have the right to step in and reschedule the debt;
6. Project CFADS paths for future periods using the distribution obtained above;
7. Determine if the debt can be refinanced: for each projected CFADS path, determine if the cash flows have transitioned into a sufficiently low risk environment where the debt can be refinanced, and determine the payout from refinancing;
8. Determine if there is a default: Compare the projected CFADS for each period with the default threshold, and if CFADS falls below the scheduled debt service, determine payout upon default;
9. Construct the cash flow waterfall with existing debt covenants: make payments according to the seniorities established in the debt contract, which would include payments to debt holders, reserve accounts, and equity holders;
10. Once cash flows to the debt holders have been projected, the present value of these cash flows is calculated under the risk-neutral probability measure using the risk-free discount rates.

3. Illustration with private infrastructure project debt

We provide a numerical implementation of our model using infrastructure project finance debt as an example.

As suggested in the introduction, infrastructure projects are typically carried out through non-recourse Project Finance (PF), which entails establishing a Special Purpose Entity (SPE) that is financed in large part by private bank loans. PF debt shares several of the key characteristics that our model aims to incorporate: 1) it is highly illiquid, with a serious lack of time series of price data, and 2) there debt contract typically contains several debt covenants (Yescombe (2002); Moody's (2015); Standard and Poor's (2013)).

Notable debt covenants include reserve accounts, cash sweeps, and step-in options. Reserve accounts reserve a certain fraction of future period's debt service. Cash sweeps are used to prepay debt if the free cash flow to equity exceeds a pre-specified threshold. And step-in option allows creditors to get involved in the firm management, restructure their debt, or take-over the firm in the case of a default event.

Exhibit 2: Flow chart for determination of cash flows to debt holders

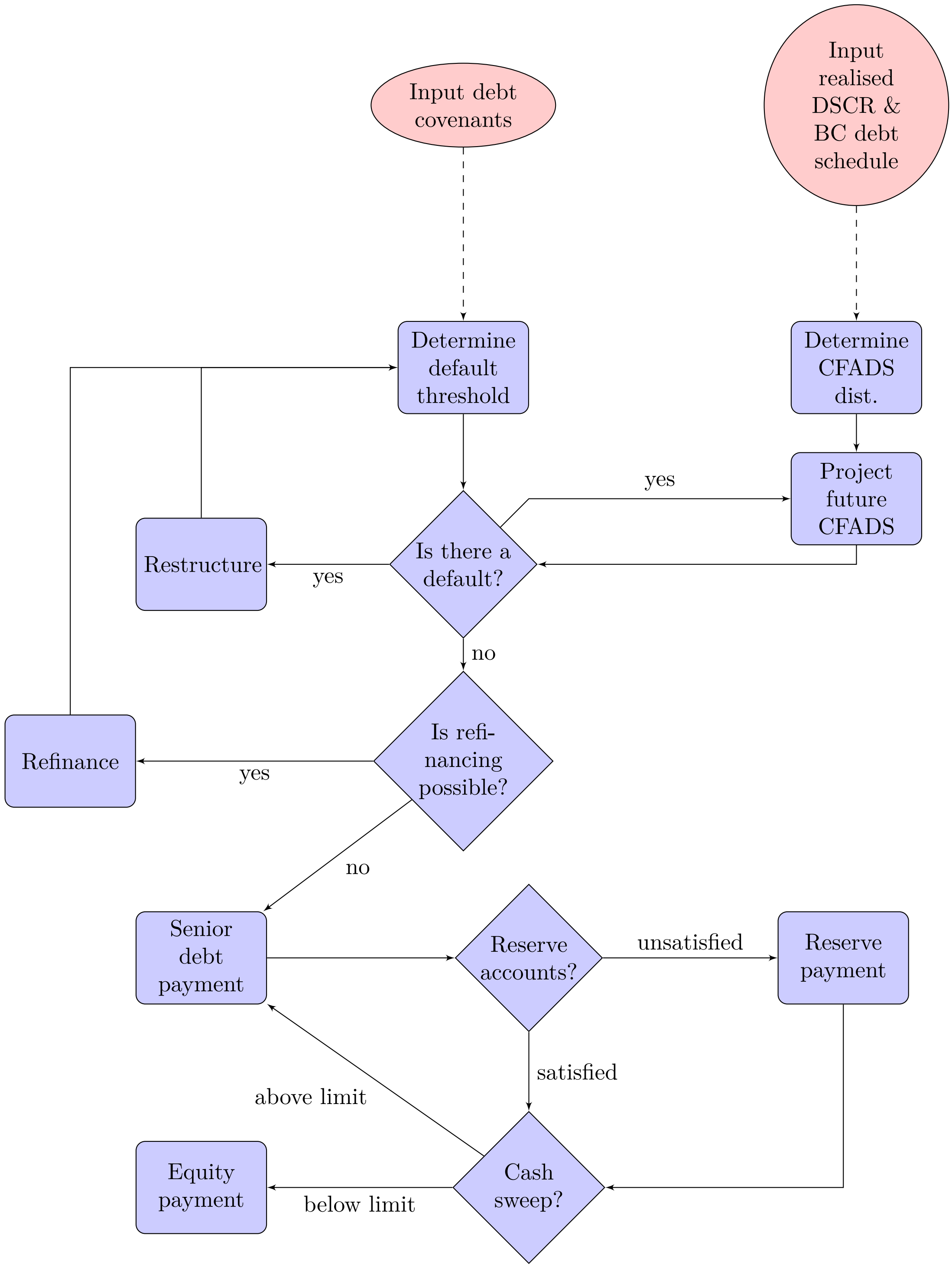


Exhibit 3: Merchant and contracted infrastructure projects

Project type	Construction period	Tail length	DSCR profile	Project maturity	First payment	Final payment	Base case IRR
Merchant	5 year	6 year	Rising	25	Year 6	Year 19	4%
Contracted	3 year	2 year	Flat	25	Year 4	Year 23	3.5%

We implement our model for two generic types infrastructure projects, each of which represents an ideal-type corresponding to numerous existing infrastructure projects. Stylised structures observed at financial close in infrastructure projects finance include the use of a rising or a flat base case DSCR profile and a more or less long "loan tail" i.e. the amount of time (and CFADS) available between the orignal debt maturity date and the end of the project.

The existence of step-in rights in combination with the debt's tail is what creates the value of the embedded option to step-in for creditors.

3.0.1. DSCR Families

A rising DSCR profile exhibits both a rising mean and implies an increasing volatility of $DSCR_t$. That is, creditors demand a higher DSCR in the future to protect themselves against rising conditional volatility of CFADS. Such projects also have longer "tails" and exhibit between 70% and 80% of initial senior leverage. Projects that are exposed to market risk, such as a power plant that sells electricity at market prices or a toll road, are structured to have a rising DSCR profile. We refer to these projects as *Merchant* infrastructure.

Conversely, a flat DSCR profile exhibits a constant mean and implies constant cash flow volatility. Projects with little to no market risk are structured with a flat DSCR. They also have shorter tails and a higher level of senior leverage, usually around 90%. Moreover, contrary to projects with a rising DSCR, which effectively de-leverage as their lifecycle unfolds, projects with a constant DSCR stay highly leveraged until the end of the debt's life. Examples of these projects include social infrastructure projects, such as schools or hospitals that receive a fixed payment from the public sector. We refer to these projects as *Contracted* infrastructure.

Exhibit 3 provides our characterisation of the two generic project structures. Both projects last for 25 year. The merchant project has a 5 year construction period, is financed with 75% leverage[6], the loan is repaid between year 6 and 19, hence a tail of 6 years.

The contracted infrastructure project has a 3 year construction period, is financed with 90% leverage, and repays the loan between years 4 and 23, leaving a tail of 2 years. Total initial debt is normalised to 1,000.

We calibrate the DSCR for the merchant project using a lognormal distribution with a constant mean return of 1%, a constant volatility of returns of 3%, an initial DSCR of 1.4, and 20% volatility of the initial DSCR.

$$\frac{d(DSCR_t)}{DSCR_t} = \mu dt + \sigma dW_t, \tag{19}$$

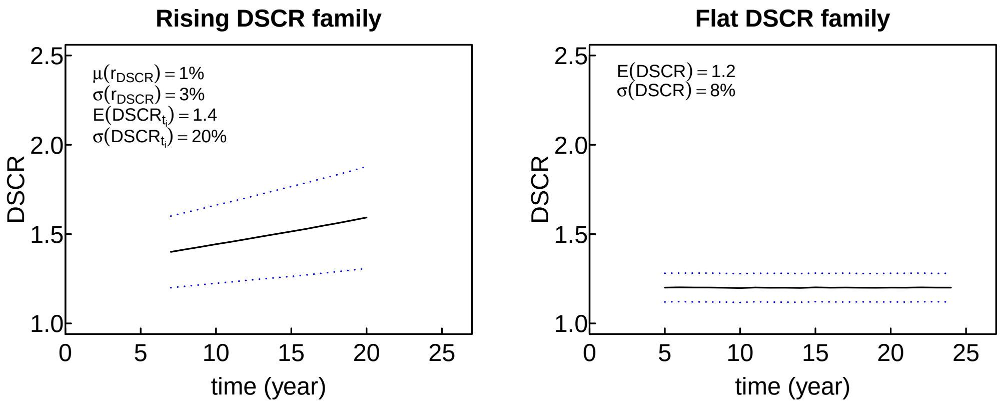
[6] We define leverage as the ratio of the market value of the loan to the market value of the SPV at financial close. Hence, the leverage is sensitive to the risk preferences of the investor. Different choices of risk preferences (Sharpe ratio) may lead to different values of debt and SPV, and hence the leverage may change. The leverage given in the table is for a benchmark investor with a Sharpe ratio of 1.

Exhibit 4: DSCR models for the two DSCR families.

DSCR profile	DSCR distribution	Mean Return	Volatility of returns	Initial expected DSCR	Volatility of initial DSCR
Rising	Lognormal	1%	3%	1.4	20%
Flat	Normal	NA	NA	1.2	8%

Exhibit 5: Physical and risk-neutral DSCR distribution for merchant and contracted infrastructure projects.

(a) DSCR models for merchant and contracted infrastructure projects.



(b) Risk neutralised DSCR distributions for the economic and social infrastructure projects.

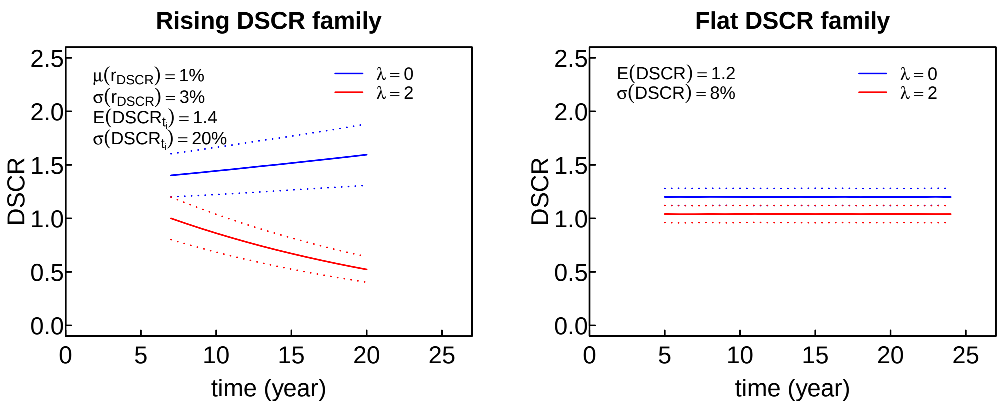
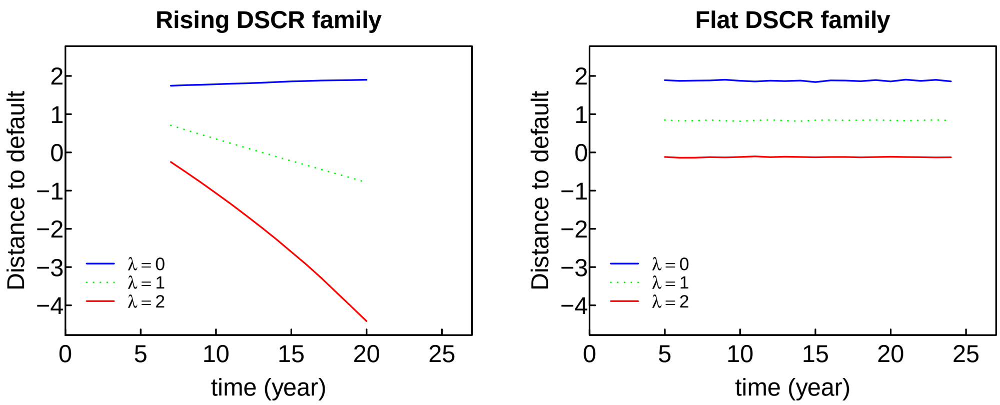
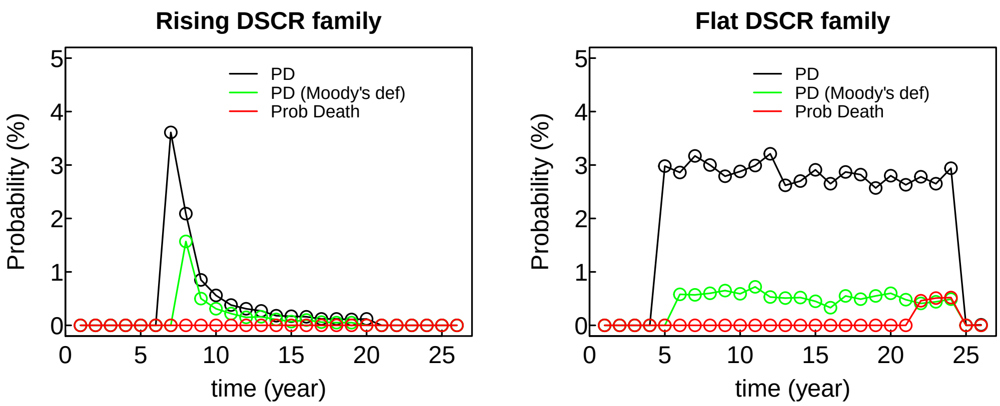


Exhibit 6: Distance to default and probabilities of default for contracted and merchant infrastructure projects.

(a) Risk neutral distance to default for the two DSCR families.



(b) Comparison of probabilities of default and death for the two DSCR families. The black line includes both technical and hard defaults. The Green line only includes hard defaults for projects that have not defaulted before, and the red line shows the probability of project company going bankrupt, i.e. the present value of its future cash flows falling below its debt value.



The DSCR for the contracted project is modelled using a normal distribution with a mean DSCR of 1.2, and a volatility of 8%.

$$DSCR_t = E[DSCR] + \sigma(DSCR)dW_t. \tag{20}$$

We list the model parameters for the two DSCR distributions in Exhibit 4. The DSCR profiles for these set of parameters are shown in Exhibit 5a.

The next step is to obtain the risk neutral distributions of the DSCR for both families, as described in Section 1.2. The risk neutral distribution for the rising DSCR family is given by

$$DSCR_t = DSCR_{t-1}e^{(\mu-\lambda\sigma-0.5\sigma^2)+\sigma dW_t}, \tag{21}$$

and the risk-neutral distribution for the flat DSCR family is given by

$$DSCR_t = E[DSCR_t] - \lambda\sigma + \sigma dW_t. \tag{22}$$

Thus, risk neutralisation effectively lowers the mean of the distribution by an amount $\lambda\sigma$, which is equivalent to discounting the cash flows, because the distribution of DSCR is directly related to the distribution of CFADS. The amount of discounting is determined by two parameters: λ and σ. λ denotes the price of risk, which is determined by the investor's risk aversion, and σ denotes the cash flow volatility. A more risk-averse investor that requires a higher price for risk, λ, would discount the cash flows more. Similarly, for a given risk-aversion, a higher volatility, σ, would make the asset more risky, and would cause investors to discount the cash flows more.

Exhibit 5b shows the risk-neutral distribution of DSCR for two choices of λ. For $\lambda = 0$, the risk-neutral distribution coincides with the physical (statistical) distribution of DSCR. Hence, $\lambda = 0$ corresponds to an investor that does not discount risky cash flows, i.e. requires no premium for bearing risk. For $\lambda = 2$, the risk neutral distribution always lies below the statistical distribution of DSCR, as a result of the higher discounting.

Exhibit 6a shows the implied distance to default for three values of λ, which correspond to different levels of risk-aversion. The higher the risk-aversion the lower the distance to default, indicating that a more risk-averse investor prices the debt as if it had a higher probability of default. Moreover, while the distance to default remains almost constant for the flat DSCR family for any level of risk aversion, the shape of the distance to default is quite sensitive to the investor's risk aversion for the rising DSCR family, and goes from upward sloping to downward sloping as the risk-aversion (λ) increases from 0 to 2. This is because in the case of flat DSCR family, the mean and volatility of the DSCR stay constant, leaving the risk-return trade-off unchanged over the life of the project. While, in contrast, in the case of rising DSCR family the mean and volatility of DSCR go up in time, changing the risk-return profile. As a result, the subjective discounting, and hence the distance to default, evolves differently over time, depending on the investor's level of risk aversion.

Finally, Exhibit 6b shows the probability of default (PD) for the two families. The PD decreases rapidly for the rising DSCR family, while it stays nearly constant for the flat DSCR family. This is because rising DSCR for merchant projects makes it unlikely for the project to default if it survives first few years post construction. While the flat DSCR for contracted projects imply that the loan is equally likely to default throughout the life of the loan. These probabilities of default are largely in line with the empirical evidence on project finance default rates reported in Moody's (2012, 2013, 2015); Standard and Poor's (2013): 1) Probability of default for contracted projects is lower compared to the merchant projects, 2) Probability of default for merchant projects goes down in time, while the probability of default for contracted projects stay roughly constant. This suggests that the observed probabilities of default in project finance can be understood directly in terms of the project's financial structuring, and most importantly its DSCR profile.

4. Extensions

Our approach allows integrating a number of observable phenomena such as the dynamics of debt service cover ratios and the terms and conditions of private debt contracts to implement a structural credit risk framework with *economically significant and empirically tractable* default thesholds defined in terms of cash flows.

It can be extended to include a model of debt restructuring post-default, a model of debt refinancing or one of the ability of the private borrower to raise new cash at different points in time. Modelling debt restructuring post-default is especially relevant if the value of creditors' option to "step-in", which we described earlier, is large and significantly impacts expected recovery rates.

5. Appendix

5.1. Relation between DSCR and CFADS Volatility

Distance to Default for infrastructure project finance loans at time t can be defined as

$$DD_t = \frac{\text{CFADS}_t - \text{DS}_t^{\text{BC}}}{\sigma_{\text{CFADS}_t}\text{CFADS}_t} \tag{23}$$

Using the definition of $DSCR_t$ in (**??**), the above expression can be written as:

$$DD_t = \frac{1}{\sigma_{\text{CFADS}_t}}(1 - \frac{1}{DSCR_t}) \tag{24}$$

The above can be re-written as a sole function of $DSCR_t$ by expressing the volatility of $CFADS_t$ as a function of that of $DSCR_t$.

We have $CFADS_t = DSCR_t \times \text{DS}_t^{\text{BC}}$, and we know that σ_{CFADS_t} is expressed as a percentage change in the asset value, thus:

$$\begin{aligned} r_{\text{CFADS}_t} &= \frac{\text{CFADS}_t}{\text{CFADS}_{t-1}} - 1 \\ &= \frac{\text{DS}_t^{\text{BC}}}{\text{DS}_{t-1}^{\text{BC}}} \frac{\text{DSCR}_t}{\text{DSCR}_{t-1}} - 1 \\ \Rightarrow \sigma_{\text{CFADS}_t} &= \sigma\left(\frac{\text{DS}_t^{\text{BC}}}{\text{DS}_{t-1}^{\text{BC}}} \frac{\text{DSCR}_t}{\text{DSCR}_{t-1}} - 1\right) \\ &= \frac{\text{DS}_t^{\text{BC}}}{\text{DS}_{t-1}^{\text{BC}}} \sigma_{\text{DSCR}_t}. \end{aligned} \tag{25}$$

Hence we can write the DD_t as

$$DD_t = \frac{1}{\sigma_{DSCR_t}} \frac{\mathrm{DS}^{\mathrm{BC}}_{t-1}}{\mathrm{DS}^{\mathrm{BC}}_{t}} (1 - \frac{1}{DSCR_t}) \tag{26}$$

where σ_{DSCR_t} is the standard deviation of the annual percentage change in the $DSCR$ value.

5.2. Decomposition of Risk Into Traded and Non-Traded Components

First, we write the current period's CFADS as

$$CFADS_{t-1} = CFADS^T_{t-1} + CFADS^N_{t-1},$$

where $CFADS^T_{t-1}$ represents the component of CFADS generated by the replicating portfolio, and $CFADS^N_{t-1}$ represents the components of the CFADS not generated by the replicating portfolio. Then, we write the mean return on CFADS as

$$\begin{aligned}
\mu &= \frac{E[CFADS_t]}{CFADS_{t-1}} - 1 \\
&= \frac{E[CFADS^T_t] + E[CFADS^N_t]}{CFADS_{t-1}} - 1 \\
&= \frac{CFADS^T_{t-1}}{CFADS_{t-1}} \frac{E[CFADS^T_t]}{CFADS^T_{t-1}} \\
&+ \frac{CFADS^N_{t-1}}{CFADS_{t-1}} \frac{E[CFADS^N_t]}{CFADS^N_{t-1}} - 1 \\
&= w^T_{t-1}(1+\mu^T) + w^N_{t-1}(1+\mu^N) - 1 \\
&= w^T_{t-1}\mu^T + w^N_{t-1}\mu^N \\
&= w^T_{t-1}(r + \lambda^T\sigma^T) + w^N_{t-1}(r + \lambda^N\sigma^N) \\
\Rightarrow \mu &= w^T_{t-1}\frac{\sigma^T}{\sigma}\lambda^T + w^N_{t-1}\frac{\sigma^N}{\sigma}\lambda^N,
\end{aligned}$$

where we have defined $w^{T(N)} = \frac{CFADS^{T(N)}_{t-1}}{CFADS_{t-1}}$, and $\lambda^{T(N)} = \frac{\mu^{T(N)} - r}{\sigma^{T(N)}}$.

6. End Notes

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